MATH 120A Prep: Equivalence Relations

Facts to Know:

Equivalence Relation: An equivalence relation $\sim$ on a set S satisfies three properties:

- Reflexive: For every $x \in S$, $x \sim x$.
- Symmetric: For every $x, y \in S$
  $x \sim y$ implies $y \sim x$
- Transitive: For every $x, y, z \in S$
  If $x \sim y$ and $y \sim z$, then $x \sim z$.

Equivalence Class: The equivalence class of an element x is

$$\{y \in S : x \sim y\} = [x]$$

* Note: equivalence classes can have different names: $x_1 \sim x_2$ then $[x_1] = [x_2]$.

Connection to Partitions:

- Equivalence Relation to Partition:
  Look at equivalence classes, they form a partition.
- Partition to Equivalence Relation:
  Starting with partition, you can define an equivalence relation where $x \sim y$ iff they reside in the same partition element

Examples:

1. Define a relation on the set of people where $A \sim B$ if the age of person A equals the age of person B. Show this is an equivalence relation.

Reflexive: If I have any person A then $A \sim A$. This means A has the same age as A.

Symmetric: If $A \sim B$, the age of A is the same as the age of B. Then B has the same age as A, and so $B \sim A$.

Transitive: If $A \sim B$ and $B \sim C$, then $A + B$ have the same age, and $B + C$ have the same age. Then $A + C$ have the same age and so $A \sim C$

Equivalence Classes

Described by the ages a person can be.

2. Show that the relation on $\mathbb{Z}$ defined by $x \sim y$ whenever $3|(x-y)$ is an equivalence relation. What are the equivalence classes?

Reflexive : $x \in \mathbb{Z}$ want $x \sim x$, so $3|(x-x)$ or $3|0$. ✓

Symmetric : Suppose $x \sim y$, so $3|(x-y)$.
Show $y \sim x$ or $3|(y-x)$. But $y-x = -1(x-y)$
So $3|y-x$ and $y \sim x$.

Transitive: Suppose $x \sim y$ and $y \sim z$, so $3|x-y$ and $3|y-z$.
Want $x \sim z$, so $3|x-z$. But $x-z = (x-y)+(y-z)$ so $3|x-z$ and $x \sim z$.

... [-2], [-1], [0], [1], [2], [3] ...

[0], [1], [2]

$x \in \mathbb{Z}$ by division algorithm $x = 3k + r$, $r \in 0,1,2$

3. Let $S = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \text{ are } 0 \text{ or } 1\}$. Define a relation on S by $(x,y,z) \sim (u,v,w)$ if $x+y+z = u+v+w$. Show this is an equivalence relation and write out the equivalence classes.

$S = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

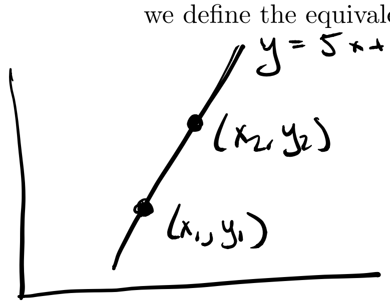
Reflexive: $(x,y,z) \sim (x,y,z)$ since $x+y+z = x+y+z$.

Symmetric: $(x,y,z) \sim (u,v,w)$ then $x+y+z = u+v+w$
so $u+v+w = x+y+z$
and $(u,v,w) \sim (x,y,z)$.

Transitive: $(x,y,z) \sim (u,v,w) \to x+y+z = u+v+w$
$(u,v,w) \sim (a,b,c) \to u+v+w = a+b+c$
$\to x+y+z = a+b+c$
$(x,y,z) \sim (a,b,c)$.

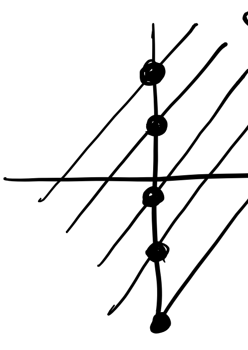
$[(0,0,0)] = \{(0,0,0)\}$: 0
$[(0,0,1)] = \{(0,0,1), (0,1,0), (1,0,0)\}$
$= [(0,1,0)] = [(1,0,0)]$: 1
$[(0,1,1)] = \{(0,1,1), (1,0,1), (1,1,0)\}$: 2
$[(1,1,1)] = \{(1,1,1)\}$: 3

4. Suppose we want an equivalence relation so that the classes are the lines of slope 5 in $\mathbb{R}^2$. How should we define the equivalence relation?



$5 = \dfrac{y_2 - y_1}{x_2 - x_1} \to y_2 - y_1 = 5(x_2 - x_1)$
$\updownarrow$
$(x_1, y_1) \sim (x_2, y_2)$

You can show this an equivalence relation.



$y = 5x + 3 \longleftrightarrow [(0,3)]$
$y = 5x + 7/2 \longleftrightarrow [(0, 7/2)]$
$y = 5x - 1/4 \longleftrightarrow [(0, -1/4)]$
$\vdots$

get a collection of equivalence classes based on the y-int.
i.e. for every value of $\mathbb{R}$ (y-int) I get a different equiv. class.